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Interval-valued fuzzy h-ideals of Γ-hemirings

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Abstract. This paper introduces the concept of interval-valued fuzzy left (resp. right) h-ideals of Γ-hemirings and investigates some related properties. It is shown that the set of all interval-valued fuzzy left h-ideals with the same tip forms a complete lattice. The characterization of h-hemiregular Γ-hemirings in terms of interval-valued fuzzy left h-ideals and interval-valued fuzzy right h-ideals is also studied.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [16]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, ect. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. On the other hand, because of the importance of group theory in mathematics as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld in [13]. Since then, there are many researchers devoting to study the relationships between fuzzy set and various algebraic structures.

Semirings which are regarded as a generalization of rings and bounded distributive lattice have been found useful in solving problems in different disciplines of applied mathematics and information sciences. It is well known that ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. In order to overcome this deficiency, Henriksen [5] defined a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring S is a ring then a complex in S is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [7]. According to Iizula's definition, an ideal in any additively commutative semiring S can be given which coincides with a ring ideal provided S is a hemiring, and it is called h-ideal. The properties of h-ideals and k-ideals of hemirings were thoroughly investigated by La Torre [10] and by using the h-ideals and k-ideals, La Torre established some analogous ring theorems for hemirings. The general properties of fuzzy h-ideals have been considered by Huang, Jun, Zhan, Yin and others. The reader is refereed to [1, 2, 3, 4, 6, 8, 9, 11, 14, 15, 18]. Ma and Zhan [12] introduced and investigated some kinds of fuzzy h-ideals of Γ-hemirings.

After the introduction of fuzzy sets by Zadeh [16], several researchers explored on the generalization of the the notion of fuzzy set. The concept of interval-valued fuzzy set was introduced by Zadeh [17] in 1975, as a generalization of the notion of fuzzy set. In this paper, we apply the concept of interval-valued fuzzy sets to Γ-hemirings. We introduce and investigate interval-valued fuzzy left (right) h-ideals. Also, we give the characterization of h-hemiregular Γ-hemirings in terms of interval-valued fuzzy left (right) h-ideals.

2. Preliminaries

In this section, we recall some basic definitions and results which will be used throughout the paper.

By an *interval number* $\tilde{a}$, we mean an interval $[a^-, a^+]$ where $0 \le a^- \le a^+ \le 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a, a]$ can be simply identified by the number a. For the interval numbers $\tilde{a}_i = [a_i^-, a_i^+]$, $\tilde{b}_i = [b_i^-, b_i^+] \in D[0,1], i \in I$, where I is an index set, we define

$\max\{\tilde{a}_i, \tilde{b}_i\} = [\max\{a_i^-, b_i^-\}, \max\{a_i^+, b_i^+\}]$,

$\min\{\tilde{a}_i, \tilde{b}_i\} = [\min\{a_i^-, b_i^-\}, \min\{a_i^+, b_i^+\}]$,

$\sup_{i\in I} \tilde{a}_i = [\sup_{i\in I} a_i^-, \sup_{i\in I} a_i^+]$,

$\inf_{i\in I} \tilde{a}_i = [\inf_{i\in I} a_i^-, \inf_{i\in I} a_i^+]$,

and put

(i) $\tilde{a} \le \tilde{b} \Leftrightarrow a^- \le b^-$ and $a^+ \le b^+$,
(ii) $\tilde{a} = \tilde{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$,
(iii) $\tilde{a} < \tilde{b} \Leftrightarrow \tilde{a} \le \tilde{b}$ and $\tilde{a} \ne \tilde{b}$,
(iv) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \le k \le 1$.

Then, it is clear that $(D[0,1], \le, \wedge, \vee)$ forms a complete lattice with $0 = [0,0]$ as its least element and $1 = [1,1]$ as its greatest element. Recall that an interval-valued fuzzy subset F on X is the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)])|x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu_F}(x)) | x \in X\}$, where $\widetilde{\mu_F} : X \to D[0,1]$. If F and G are two interval-valued fuzzy subsets of X, then we define

$F \subseteq G$ if and only if for all $x \in X$, $\mu_F^-(x) \leq \mu_G^-(x)$ and $\mu_F^+(x) \leq \mu_G^+(x)$,

$F = G$ if and only if for all $x \in X$, $\mu_F^-(x) = \mu_G^-(x)$ and $\mu_F^+(x) = \mu_G^+(x)$.

Also, the *union, intersection* and *complement* are defined as follows:

$F \cup G = \{(x, [\max\{\mu_F^-(x), \mu_G^-(x)\}, \max\{\mu_F^+(x), \mu_G^+(x)\}]) | x \in X\}$,

$F \cap G = \{(x, [\min\{\mu_F^-(x), \mu_G^-(x)\}, \min\{\mu_F^+(x), \mu_G^+(x)\}]) | x \in X\}$.

Definition 2.1. [12] Let S and Γ be two additive abelian groups. S is called a Γ-*semiring* if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfies the following conditions:

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$;
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$;
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$;
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a *zero* of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with a zero is said to be a Γ-*hemiring*.

Throughout this paper, S is a Γ-hemiring, and we use 0_S to denote the zero element of S.

A *left* (resp., *right*) *ideal* of a Γ-hemiring S is a subset A of S which is closed under addition such that $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$), where $S\Gamma A = \{a\Gamma b | a, b \in S, \alpha \in \Gamma\}$. Further, A is called an *ideal* of S if it is both a left ideal and a right ideal of S.

A *left ideal* (*right ideal, ideal*) A of S is called a *left h-ideal* (*right h-ideal, h-ideal*) of S, respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z \to x \in A$.

The *h-closure* $\overline{A}$ of A in S is defined by

$$\overline{A} = \{x | x + a + z = b + z \text{ for some } a, b \in A, z \in S\}.$$

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $A \subseteq \overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

We introduce the h-sum and h-product of two interval-valued fuzzy subsets as follows.

Definition 2.2. Let F and G be two interval-valued fuzzy subsets of S. The *h-sum* of F and G, denoted by $F +_h G$, is defined by

$$\widetilde{\mu_{F+_hG}}(x) = \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\}.$$

Definition 2.3. Let F and G be two interval-valued fuzzy subsets of S. Then the *h-product* of F and G, denoted by $F\Gamma_h G$, is defined by

$$\widetilde{\mu_{F\Gamma_hG}}(x) = \bigvee_{x+a_1\gamma_1b_1+z=a_2\gamma_2b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\}$$

and $\widetilde{\mu_{F\Gamma_hG}}(x) = 0$ if x can not be expressed as $x + a_1\gamma_1b_1 + z = a_2\gamma_2b_2 + z$.

By directly calculation we obtain immediately the following results.

Lemma 2.4. *Let G_1, G_2, H_1 and H_2 be interval-valued fuzzy subsets of S such that $G_1 \subseteq H_1$ and $G_2 \subseteq H_2$. Then $G_1 +_h H_1 \subseteq G_2 +_h H_2$ and $G_1 \Gamma_h H_1 \subseteq G_2 \Gamma_h H_2$.*

Lemma 2.5. *Let F, G and H be interval-valued fuzzy subsets of a hemiring S. Then we have*

(1) $F +_h (H \cup G) = (F \cup H) +_h (F \cup G)$ *and* $F\Gamma_h(H \cup G) = (F \cup H)\Gamma_h(F \cup G)$;
(2) $F +_h (H \cap G) \subseteq (F \cap H) +_h (F \cap G)$ *and* $F\Gamma_h(H \cap G) \subseteq (F \cap H)\Gamma_h(F \cap G)$.

For any non-empty subset A of S, we denote by $\chi_A = \{(x, \widetilde{\mu_{\chi_A}}(x)) | x \in S\}$, where $\widetilde{\mu_{\chi_A}}(x)$ is defined by $\widetilde{\mu_{\chi_A}}(x) = 1$ if $x \in A$ and otherwise $\widetilde{\mu_{\chi_A}}(x) = 0$.

Lemma 2.6. *Let $A, B \subseteq S$. Then we have*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \chi_B$;
(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$;
(3) $\chi_A +_h \chi_B = \chi_{\overline{A+B}}$.
(4) $\chi_A \Gamma_h \chi_B = \chi_{\overline{A\Gamma B}}$.

Proof. We only show (4). Let $x \in S$. If $\widetilde{\mu_{\chi_{\overline{A\Gamma B}}}}(x) = 1$, then $x \in \overline{A\Gamma B}$ and so there exist $a_1', a_2' \in A, b_1', b_2' \in B, \gamma_1', \gamma_2' \in \Gamma$ and $z' \in S$ such that $x' + a_1'\gamma_1'b_1' + z = a_2'\gamma_2'b_2' + z'$. Hence

$$\widetilde{\mu_{\chi_A \Gamma_h \chi_B}}(x) = \bigvee_{x + a_1\gamma_1 b_1 + z = a_2\gamma_2 b_2 + z} \min\{\widetilde{\mu_{\chi_A}}(a_1), \widetilde{\mu_{\chi_A}}(a_2), \widetilde{\mu_{\chi_B}}(b_1), \widetilde{\mu_{\chi_B}}(b_2)\}$$
$$\geq \min\{\widetilde{\mu_{\chi_A}}(a_1'), \widetilde{\mu_{\chi_A}}(a_2'), \widetilde{\mu_{\chi_B}}(b_1'), \widetilde{\mu_{\chi_B}}(b_2')\} = 1.$$

Otherwise, if $\widetilde{\mu_{\chi_{\overline{A\Gamma B}}}}(x) = 0$, then $x \notin \overline{A\Gamma B}$, which implies that x cannot be expressed as $x + a_1\gamma_1 b_1 + z = a_2\gamma_2 b_2 + z$ and so $\widetilde{\mu_{\chi_A \Gamma_h \chi_B}}(x) = 0$. Therefore, in any case we have $\widetilde{\mu_{\chi_A \Gamma_h \chi_B}}(x) = \widetilde{\mu_{\chi_{\overline{A\Gamma B}}}}(x)$ and so $\chi_A \Gamma_h \chi_B = \chi_{\overline{A\Gamma B}}$. □

3. The interval-valued fuzzy left (right) h-ideals of Γ-hemirings

Definition 3.1. An interval-valued fuzzy subset F of S is called an *interval-valued fuzzy h-left ideal* if for all $x, z, a, b \in S$ we have

(F1a) $F +_h F \subseteq F$,
(F2a) $\chi_S \Gamma_h F \subseteq F$,
(F3a) $x + a + z = b + z \rightarrow \widetilde{\mu_F}(x) \geq \min\{\widetilde{\mu_F}(a), \widetilde{\mu_F}(b)\}$.

Interval-valued fuzzy right h-ideals and *interval-valued fuzzy h-ideals* are defined similarly.

Note that for any interval-valued fuzzy left (right) h-ideal F of S, $\widetilde{\mu_F}(0) \geq \widetilde{\mu_F}(x)$ for all $x \in S$ since $0 + x + z = x + z$.

Lemma 3.2. *Let F be an interval-valued fuzzy subset of S such that* (F3a) *holds. Then* (F1a) *holds if and only if the following condition holds:*

(F1b) $\widetilde{\mu_F}(x + y) \geq \min\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$ *for all $x, y \in S$.*

Proof. Assume that (F1a) holds. Let $x, y \in S$. Then, since $x+y+0+0+z = x+y+z$ for any $z \in S$, we have

$$\begin{aligned}\widetilde{\mu_F}(x+y) &\geq \widetilde{\mu_{F+_hF}}(x+y)\\ &= \bigvee_{x+y+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_F}(b_1), \widetilde{\mu_F}(b_2)\}\\ &\geq \min\{\widetilde{\mu_F}(0), \widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geq \min\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}.\end{aligned}$$

Conversely, assume that (F1b) holds. Let $x \in S$. Then $\widetilde{\mu_{F+_hF}}(x) = 0 \leq \widetilde{\mu_F}(x)$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$. Otherwise, by (F3a) and (F1b), we have

$$\widetilde{\mu_F}(x) \geq \min\{\widetilde{\mu_F}(a_1+b_1), \widetilde{\mu_F}(a_2+b_2)\} \geq \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_F}(b_1), \widetilde{\mu_F}(b_2)\}$$

and so

$$\begin{aligned}\widetilde{\mu_{F\Gamma_hF}}(x) &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_F}(b_1), \widetilde{\mu_F}(b_2)\}\\ &\leq \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \widetilde{\mu_F}(x) = \widetilde{\mu_F}(x).\end{aligned}$$

Hence $\chi_S +_h F \subseteq F$. □

Lemma 3.3. *Let F be an interval-valued fuzzy subset of S such that* (F3a) *holds. Then $(F2a)$ holds if and only if the following condition holds:*
(F2b) $\widetilde{\mu_F}(x\alpha y) \geq \widetilde{\mu_F}(y)$ *for all $x, y \in S$ and $\alpha \in \Gamma$.*

Proof. The proof is similar to that of Lemma 3.2. □

By Lemmas 3.2 and 3.3, we have the following results.

Theorem 3.4. *An interval-valued fuzzy subset F of S is an interval-valued fuzzy h-ideal of S if it satisfies* (F1b), (F3a) *and*
(F3b) $\widetilde{\mu_F}(x\alpha y) \geq \max\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$ *for all $x, y \in S$ and $\alpha \in \Gamma$.*

Let F be an interval-valued fuzzy subset of S. Then, for every $\widetilde{a} \in D[0,1]$, the sets $U(F, \tilde{a}) = \{x \in S | \widetilde{\mu_F}(x) \geq \tilde{a}\}$ and $\overline{U}(F, \tilde{a}) = \{x \in S | \widetilde{\mu_F}(x) > \tilde{a}\}$ are called an *interval-valued level subset* and an *interval-valued strong level subset* of F, respectively. The following lemma indicates the relationships between interval-valued fuzzy left (resp. right) h-ideals of S and crisp left (resp. right) h-ideals of S.

Lemma 3.5. *Let F be an interval-valued fuzzy subset of S. Then we have*

(1) *F is an interval-valued fuzzy left (resp. right) h-ideal of S if and only if $U(F, \tilde{a})(U(F, \tilde{a}) \neq \varnothing)$ is a left (resp. right) h-ideal of S for all $\widetilde{a} \in D[0,1]$.*
(2) *F is an interval-valued fuzzy left (resp. right) h-ideal of S if and only if $\overline{U}(F, \tilde{a})(\overline{U}(F, \tilde{a}) \neq \varnothing)$ is a left (resp. right) h-ideal of S for all $\widetilde{a} \in D[0,1]$.*

Proof. The proof is straightforward. □

As a direct consequence, we have the following result.

Lemma 3.6. *Let $A \subseteq S$. Then A is a left (resp. right) h-ideal of S if and only if χ_A is an interval-valued fuzzy left (resp. right) h-ideal of S.*

Theorem 3.7. *Let F and G be interval-valued fuzzy left (resp. right) h-ideals of S. Then so is $F \cap G$.*

Proof. Let F and G be two interval-valued fuzzy left h-ideals of S. We show that $F \cap G$ is also an interval-valued fuzzy left h-ideal of S. The case for interval-valued fuzzy right h-ideals can be similarly proved.

(1) By Lemma 2.4, $(F \cap G) +_h (F \cap G) \subseteq F +_h F \subseteq F$ and $(F \cap G) +_h (F \cap G) \subseteq G +_h G \subseteq G$. Hence $(F \cap G) +_h (F \cap G) \subseteq F \cap G$.

(2) By Lemma 2.4, $\chi_S \Gamma_h (F \cap G) \subseteq \chi_S \Gamma_h F \subseteq F$ and $\chi_S \Gamma_h (F \cap G) \subseteq \chi_S \Gamma_h G \subseteq G$. Hence $\chi_S \Gamma_h (F \cap G) \subseteq F \cap G$.

(3) Let a, b, x and z be any elements of S such that $x + a + z = b + z$. Then it follows that

$$\begin{aligned}\widetilde{\mu_{F\cap G}}(x) &= \min\{\widetilde{\mu_F}(x), \widetilde{\mu_G}(x)\} \geq \min\{\widetilde{\mu_F}(a), \widetilde{\mu_F}(b), \widetilde{\mu_G}(a), \widetilde{\mu_G}(b)\} \\ &= \min\{\widetilde{\mu_{F\cap G}}(a), \widetilde{\mu_{F\cap G}}(b)\}.\end{aligned}$$

Therefore, $F \cap G$ is an interval-valued fuzzy left h-ideal of S. □

Theorem 3.8. *Let F and G be interval-valued fuzzy left (resp. right) h-ideals of S. Then so is $F +_h G$.*

Proof. Let F and G be interval-valued fuzzy left h-ideals of S. We show that $F +_h G$ is also an interval-valued fuzzy left h-ideal of S. The case for interval-valued fuzzy right h-ideals can be similarly proved.

(1) For any $x, y \in S$, we have

$$\begin{aligned}\widetilde{\mu_{F+_hG}}(x+y) &= \bigvee_{x+y+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\} \\ &\geq \min\left\{ \bigvee_{x+c_1+d_1+z_1=c_2+d_2+z_1} \min\{\widetilde{\mu_F}(c_1), \widetilde{\mu_F}(c_2), \widetilde{\mu_G}(d_1), \widetilde{\mu_G}(d_2)\},\right. \\ &\qquad \left. \bigvee_{y+e_1+f_1+z_2=e_2+f_2+z_2} \min\{\widetilde{\mu_F}(e_1), \widetilde{\mu_F}(e_2), \widetilde{\mu_G}(f_1), \widetilde{\mu_G}(f_2)\} \right\} \\ &= \min\{\widetilde{\mu_{F+_hG}}(x), \widetilde{\mu_{F+_hG}}(y)\}.\end{aligned}$$

(2) For any $x, y \in S$ and $\alpha \in \Gamma$, we have

$$\begin{aligned}\widetilde{\mu_{F+_hG}}(y) &= \bigvee_{y+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\} \\ &\leq \bigvee_{\substack{x\alpha y+x\alpha a_1+x\alpha b_1+x\alpha z \\ =x\alpha a_2+x\alpha b_2+x\alpha z}} \min\{\widetilde{\mu_F}(x\alpha a_1), \widetilde{\mu_F}(x\alpha a_2), \widetilde{\mu_G}(x\alpha b_1), \widetilde{\mu_G}(x\alpha b_2)\} \\ &\leq \bigvee_{x\alpha y+c_1+d_1+z_1=c_2+d_2+z_1} \min\{\widetilde{\mu_F}(c_1), \widetilde{\mu_F}(c_2), \widetilde{\mu_G}(d_1), \widetilde{\mu_G}(d_2)\} \\ &= \widetilde{\mu_{F+_hG}}(x\alpha y).\end{aligned}$$

(3) Let a, b, x and z_1 be any elements of S such that $x + a + z_1 = b + z_1$. If there exist $c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, z_2, z_3 \in S$ such that

$a + c_1 + d_1 + z_2 = c_2 + d_2 + z_2$ and $b + e_1 + f_1 + z_3 = e_2 + f_2 + z_3$,
then we have

$$x + c_2 + d_2 + e_1 + f_1 + z_4 = c_1 + d_1 + e_2 + f_2 + z_4,$$

where $z_4 = z_1 + z_2 + z_3$, and so

$$\begin{aligned}\widetilde{\mu_{F+_hG}}(x) &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\}\\ &\geq \min\{\widetilde{\mu_F}(c_2 + e_1), \widetilde{\mu_F}(c_1 + e_2), \widetilde{\mu_G}(d_2 + f_1), \widetilde{\mu_G}(d_1 + f_2)\}\\ &\geq \min\{\min\{\widetilde{\mu_F}(c_2), \widetilde{\mu_F}(e_1)\}, \min\{\widetilde{\mu_F}(c_1), \widetilde{\mu_F}(e_2)\},\\ &\qquad \min\{\widetilde{\mu_G}(d_2), \widetilde{\mu_G}(f_1)\}, \min\{\widetilde{\mu_G}(d_1), \widetilde{\mu_G}(f_2)\}\}\\ &= \min\{\widetilde{\mu_F}(c_2), \widetilde{\mu_F}(e_1), \widetilde{\mu_F}(c_1), \widetilde{\mu_F}(e_2), \widetilde{\mu_G}(d_2), \widetilde{\mu_G}(f_1), \widetilde{\mu_G}(d_1), \widetilde{\mu_G}(f_2)\}.\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{\mu_{F+_hG}}(x) \geq \min\Bigg\{ &\bigvee_{a+c_1+d_1+z_2=c_2+d_2+z_2} \min\{\widetilde{\mu_F}(c_1), \widetilde{\mu_F}(c_2), \widetilde{\mu_G}(d_1), \widetilde{\mu_G}(d_2)\},\\ &\bigvee_{b+e_1+f_1+z_3=e_2+f_2+z_3} \min\{\widetilde{\mu_F}(e_1), \widetilde{\mu_F}(e_2), \widetilde{\mu_G}(f_1), \widetilde{\mu_G}(f_2)\}\Bigg\}\\ = \min\{&\widetilde{\mu_{F+_hG}}(a), \widetilde{\mu_{F+_hG}}(b)\}.\end{aligned}$$

Therefore, $F +_h G$ is an interval-valued fuzzy left h-ideal of S. □

Denote by $\mathbb{IVFHI}(S)$ the set of all interval-valued fuzzy left h-ideals of S with the same tip, that is, $\widetilde{\mu_F}(0) = \widetilde{\mu_G}(0)$ for all $F, G \in \mathbb{IVFHI}(S)$.

Theorem 3.9. $(\mathbb{IVFHI}(S), \cap, +_h)$ *is a complete lattice.*

Proof. Let $F, G \in (\mathbb{IVFHI}(S), \cap, +_h)$. Clearly, $F \cap G \in \mathbb{IVFHI}(S)$ is the greatest lower bound of F and G. By Theorem 3.8, we have $F +_h G \in \mathbb{IVFHI}(S)(S)$. Now, we prove that $F +_h G$ is the least supper bound of F and G. Let $x \in S$. Since $x + 0 + 0 + 0 = x + 0 + 0$, we have

$$\begin{aligned}\widetilde{\mu_{F+_hG}}(x) &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\}\\ &\geq \min\{\widetilde{\mu_F}(0), \widetilde{\mu_F}(x), \widetilde{\mu_G}(0)\} = \widetilde{\mu_F}(x), \widetilde{\mu_F}(0)\}\\ &\geq \widetilde{\mu_F}(x).\end{aligned}$$

It follows that $F \subseteq F +_h G$. Similarly, $G \subseteq F +_h G$ and so $A +_h B$ is an upper bound of F and G. Now, let $H \in \mathbb{IVFHI}(S)$ be such that $F \subseteq H$ and $G \subseteq H$. It follows from Lemma 2.4 that $F +_h G \subseteq C +_h C \subseteq C$. So, $F +_h G$ is the least supper bound of F and G. There is no difficulty in replacing the $\{F, G\}$ with an arbitrary family of $\mathbb{IVFHI}(S)$ and so $(\mathbb{IVFHI}(S), \cap, +_h)$ is complete. Therefore, $(\mathbb{IVFHI}(S), \cap, +_h)$ is a complete lattice. □

4. The characterization of h-hemiregular Γ-hemirings

In this section, we consider the characterization of h-hemiregular Γ-hemirings in terms of an interval-valued fuzzy left h-ideal and an interval-valued fuzzy right h-ideal.

Definition 4.1. A Γ-hemiring S is said to be *h-hemiregular* if for each $x \in S$, there exist $a, a', z \in S$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$ such that

$$x + x\alpha a\beta x + z = x\alpha' a'\beta' x + z.$$

Lemma 4.2. [12] *A Γ-hemiring S is h-hemiregular if and only if for every right h-ideal R and every left h-ideal L of S we have $\overline{R\Gamma_h L} = R \cap L$.*

Theorem 4.3. *A Γ-hemiring S is h-hemiregular if and only if for every interval-valued fuzzy right h-ideal F and every interval-valued fuzzy left h-ideal G of S we have $F\Gamma_h G = F \cap G$.*

Proof. Assume that S is h-hemiregular. Let F and G be any interval-valued fuzzy right h-ideal and any interval-valued fuzzy left h-ideal of S, respectively. Then by Lemma 2.4, we have $F\Gamma_h G \subseteq F\Gamma_h \chi_S \subseteq F$ and $F\Gamma_h G \subseteq \chi_S \Gamma_h G \subseteq G$. Thus $F\Gamma_h G \subseteq F \cap G$. To show the converse inclusion, for any $x \in S$, since S is h-hemiregular, there exist $a, a', z \in S$ and $\alpha, \beta, \alpha', \beta' \in \Gamma$ such that $x + x\alpha a\beta x + z = x\alpha' a'\beta' x + z$. Then we have

$$\begin{aligned}\widetilde{\mu_{F\Gamma_h G}}(x) &= \bigvee_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{\widetilde{\mu_F}(a_1), \widetilde{\mu_F}(a_2), \widetilde{\mu_G}(b_1), \widetilde{\mu_G}(b_2)\}\\ &\geq \min\{\widetilde{\mu_F}(x\alpha a), \widetilde{\mu_F}(x\alpha a'), \widetilde{\mu_G}(x)\}\\ &\geq \min\{\widetilde{\mu_F}(x), \widetilde{\mu_G}(x)\} = (\widetilde{\mu_F} \cap \widetilde{\mu_G})(x),\end{aligned}$$

this implies that $F \cap G \subseteq F\Gamma_h G$. Therefore $F\Gamma_h G = F \cap G$.

Conversely, assume that the given condition holds. Let L and R be any left h-ideal and any right h-ideal of S, respectively. Then by Lemma 3.6, χ_L and χ_R is an interval-valued fuzzy left h-ideal and an interval valued fuzzy right h-ideal of S, respectively. Then, by the assumption and Lemma 2.6, we have

$$\chi_{R\cap L} = \chi_R \cap \chi_L = \chi_R \Gamma_h \chi_L = \chi_{\overline{L\Gamma R}}$$

and so it follows from Lemma 2.6 that $R \cap L = \overline{R\Gamma L}$. Hence S is h-hemiregular by Lemma 4.2. ☐

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